JAL C12 May 16 Kprime 2

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1. The first three terms in ascending powers of x in the binomial expansion of $(1 + px)^8$ are given by

$$1 + 12x + qx^2$$

where p and q are constants.

Find the value of p and the value of q.

(5)

$(1+px)^8 = 1+12x+qx^2+\dots$

LHS $= 1^8 + 8(1)^7(px) + 28(1)^6p^2x^2 \dots$

$\therefore \quad 8p=12 \Rightarrow p = \frac{3}{2}$

$28p^2 = q \Rightarrow q = 63$

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2. Find the range of values of x for which

(a) $4(x-2) \leqslant 2x+1$ **(2)**

(b) $(2x-3)(x+5) > 0$ **(3)**

(c) **both** $4(x-2) \leqslant 2x+1$ **and** $(2x-3)(x+5) > 0$ **(1)**

(a). $4x - 8 \leq 2x + 1$

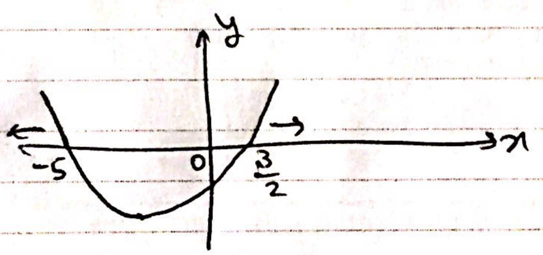
$\therefore \quad 2x \leq 9$

$\therefore \quad x \leq 9/2$

~~(b) $(2x-3)(x+5) = 2x^2 + 7x - 15$~~

~~$\therefore 2x^2 + 7x$~~

(b)



$x = \frac{3}{2}$

$x = -5$

$\therefore \quad x > \frac{3}{2} \qquad x < -5$

(c) $\left\{\frac{3}{2} < x \leq \frac{9}{2}\right\} \cup \{x < -5\}$

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3. Answer this question without a calculator, showing all your working and giving your answers in their simplest form.

(i) Solve the equation

$$4^{2x+1} = 8^{4x}$$

(3)

(ii) (a) Express

$$3\sqrt{18} - \sqrt{32}$$

in the form $k\sqrt{2}$, where k is an integer.

(2)

(b) Hence, or otherwise, solve

$$3\sqrt{18} - \sqrt{32} = \sqrt{n}$$

(2)

(i) $4^{2x+1} = 8^{4x}$

$\therefore (2^2)^{2x+1} = (2^3)^{4x}$

$\therefore 2^{4x+2} = 2^{12x}$

$\Rightarrow 4x + 2 = 12x \Rightarrow 8x = 2 \therefore x = \frac{1}{4}$

(ii) (a). $3\sqrt{18} - \sqrt{32} = 3\sqrt{9}\sqrt{2} - \sqrt{16}\sqrt{2}$

$= 9\sqrt{2} - 4\sqrt{2}$

$\therefore 3\sqrt{18} - \sqrt{32} = 5\sqrt{2}$ $k = 5$

(c) $5\sqrt{2} = \sqrt{n}$

$(5\sqrt{2})^2 = (\sqrt{n})^2 \Rightarrow n = 50$

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4.

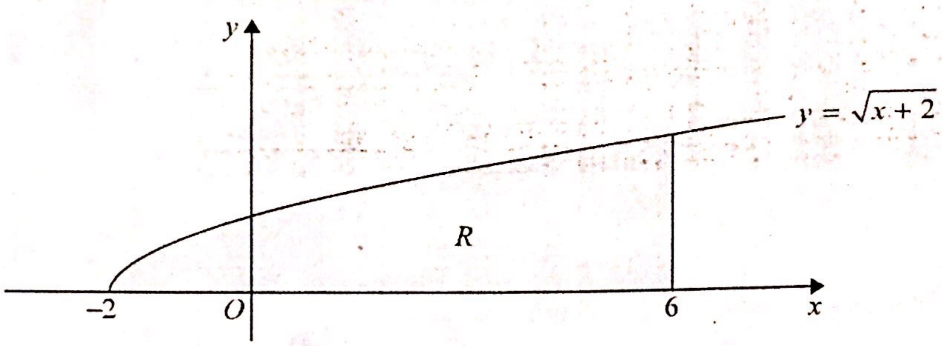


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \sqrt{x+2},\ x \geqslant -2$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the line $x = 6$

The table below shows corresponding values of x and y for $y = \sqrt{x+2}$

x	−2	0	2	4	6
y	0	1.4142	2	2.4495	2.8284

(a) Complete the table above, giving the missing value of y to 4 decimal places.

(1)

(b) Use the trapezium rule, with all of the values of y in the completed table, to find an approximate value for the area of R, giving your answer to 3 decimal places.

(3)

Use your answer to part (b) to find approximate values of

(c) (i) $\displaystyle\int_{-2}^{6} \frac{\sqrt{x+2}}{2}\,dx$

(ii) $\displaystyle\int_{-2}^{6} \left(2 + \sqrt{x+2}\right) dx$

(4)

(b) Area R $\approx \left(\cancel{\frac{1}{2}}\right)\left(\cancel{\frac{8}{4}}\right)[0 + 2(1.4142 + 2 + 2.4495) + 2.8284]$

$= 14.556$ (3dp)

P46713RA0852

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA (c)(i) $\int_{-2}^{6} \frac{\sqrt{x+2}}{2}\,dx = \frac{1}{2}\int_{-2}^{6}\sqrt{x+2}\,dx \approx \frac{1}{2} \times 14.556$

$= 7.278$

(ii) $\int_{-2}^{6} 2+\sqrt{x+2}\,dx = \int_{-2}^{6} 2\,dx + \int_{-2}^{6}\sqrt{x+2}\,dx$

$\approx [2x]_{-2}^{6} + 14.556$

$= 16 + 14.556 = 30.556$

5. (i) $$U_{n+1} = \frac{U_n}{U_n - 3}, \quad n \geqslant 1$$

Given $U_1 = 4$, find

(a) U_2 **(1)**

(b) $\sum_{n=1}^{100} U_n$ **(2)**

(ii) Given

$$\sum_{r=1}^{n} (100 - 3r) < 0$$

find the least value of the positive integer n. **(3)**

5(i)(a) $U_2 = \frac{4}{4-3} = 4$

(b) $U_1 = U_2 = U_3 = U_4 = U_5 = \ldots = U_n = 4$

$\sum_{n=1}^{100} U_n = \sum_{n=1}^{100} 4 = 4 \times 100 = 400$

(ii) $\sum_{r=1}^{n} 100-3r = \sum_{r=1}^{n} 100 - 3\sum_{r=1}^{n} r$

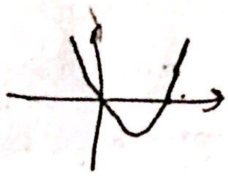
$= 100n - \frac{3n}{2}(n+1)$

$\therefore 100n - \frac{3n}{2}(n+1) < 0$

$\therefore 100n - \frac{3}{2}n^2 - \frac{3n}{2} < 0$

$200n - 3n^2 - 3n < 0$

$\therefore \quad 3n^2 - 197n > 0$



$n(3n - 197) > 0$

C.V $\quad n = \frac{197}{3} = 65.66\ldots$

$\Rightarrow n = 66$

6. (a) Show that $\frac{x^2-4}{2\sqrt{x}}$ can be written in the form $Ax^p + Bx^q$, where A, B, p and q are constants to be determined. **(3)**

(b) Hence find

$$\int \frac{x^2-4}{2\sqrt{x}}\,dx, \quad x > 0$$

giving your answer in its simplest form. **(4)**

~~6(a)~~ $\cancel{\frac{x^2-4}{2\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}} = x^{5/2} - 4x^{1/2}}$

(a) $\frac{x^2-4}{2\sqrt{x}} = \frac{x^2}{2x^{1/2}} - \frac{4}{2\sqrt{x}}$

$\cancel{= \frac{1}{2}x^{3/2} - \frac{2}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}\times\sqrt{x}}}$

$= \frac{1}{2}x^{3/2} - 2x^{-1/2}$ $\quad A = \frac{1}{2} \quad B = -2$

$p = 3/2 \quad q = -\frac{1}{2}$

(b) $\int \frac{1}{2}x^{3/2} - 2x^{-1/2}\, dx$

$= \frac{1}{5}x^{5/2} - 4x^{1/2} + C$

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7.

$$f(x) = 3x^3 + ax^2 + bx - 10, \text{ where } a \text{ and } b \text{ are constants.}$$

Given that $(x - 2)$ is a factor of $f(x)$,

(a) use the factor theorem to show that $2a + b = -7$ **(2)**

Given also that when $f(x)$ is divided by $(x + 1)$ the remainder is -36

(b) find the value of a and the value of b. **(4)**

$f(x)$ can be written in the form

$$f(x) = (x - 2)Q(x), \text{ where } Q(x) \text{ is a quadratic function.}$$

(c) (i) Find $Q(x)$.

(ii) Prove that the equation $f(x) = 0$ has only one real root.

You must justify your answer and show all your working. **(4)**

7

(a) $(x-2)$ as a factor implies $x=2$ is a root of $f(x)=0$

$\therefore f(2) = 3(2)^3 + a(2)^2 + 2b - 10 = 0$

$f(2) = 24 + 4a + 2b - 10$

$= 4a + 2b + 14 = 0$

$\therefore 4a + 2b = -14$

$\div 2 \Rightarrow 2a + b = -7$ #

(b) $f(-1) = -36$

$f(-1) =$

$\Rightarrow -3 + a - b - 10 = -36$

$\therefore a - b = -23$

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$2a+b+a-b = 3a = -7-23 = -30$

$\therefore \quad a = -10$

$\Rightarrow \quad b = a+23 = -10+23 = 13$

$a = -10$
$b = 13$

(c) (i)

$3x^3 - 10x^2 + 13x - 10 \equiv (x-2)[3x^2 + kx + 5]$

$\therefore 3x^3 - 10x^2 + 13x - 10 = 3x^3 + kx^2 + 5x - 6x^2 - 2kx - 10$

$= 3x^3 + (k-6)x^2 + (5-2k)x - 10$

$\Rightarrow \quad k - 6 = -10$
$\Rightarrow k = -4$

$\therefore \quad Q(x) = 3x^2 - 4x + 5$

(ii) $f(x) = 0 \Rightarrow (x-2)(3x^2 - 4x + 5) = 0$

$\Rightarrow \quad x - 2 = 0 \quad \therefore \quad x = 2$

$Q(x) = 0$ Discriminant of $Q(x) = b^2 - 4ac = (-4)^2 - 4(3)(5)$
$= -44 < 0$

$\therefore \quad b^2 - 4ac < 0 \quad \Rightarrow Q(x)$ has no real roots.

$\therefore x = 2$ is the only root

8. In this question the angle θ is measured in degrees throughout.

(a) Show that the equation

$$\frac{5+\sin\theta}{3\cos\theta} = 2\cos\theta, \qquad \theta \neq (2n+1)90°, \quad n \in \mathbb{Z}$$

may be rewritten as

$$6\sin^2\theta + \sin\theta - 1 = 0$$

(3)

(b) Hence solve, for $-90° < \theta < 90°$, the equation

$$\frac{5+\sin\theta}{3\cos\theta} = 2\cos\theta$$

Give your answers to one decimal place, where appropriate.

(4)

8(a). $\frac{5+\sin\theta}{3\cos\theta} = 2\cos\theta$

$\times\ 3\cos\theta \Rightarrow 5+\sin\theta = 6\cos^2\theta$

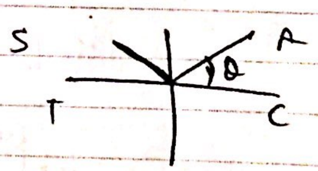
$\cos^2\theta = 1-\sin^2\theta$

$\therefore 5+\sin\theta = 6-6\sin^2\theta$

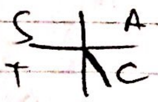
$\therefore \quad 6\sin^2\theta + \sin\theta - 1 = 0$

(b) $(3\sin\theta - 1)(2\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{3} \Rightarrow \theta = 19.5°$ (1dp)



$\sin\theta = -\frac{1}{2}$



$\theta = -30°$

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9. The first term of a geometric series is 6 and the common ratio is 0.92

For this series, find

(a) (i) the 25th term, giving your answer to 2 significant figures,

(ii) the sum to infinity. **(4)**

The sum to n terms of this series is greater than 72

(b) Calculate the smallest possible value of n. **(4)**

9(a) $a = 6 \qquad r = 0.92$

(i) $u_{25} = 6 \times 0.92^{24}$

$= 0.81$ (2sf)

(ii) $S_\infty = \dfrac{6}{1-0.92} = 75$

(b) $S_n > 72$

$\therefore \dfrac{6(1-0.92^n)}{1-0.92} > 72$

$6(1-0.92^n) > 5.76$

$1 - 0.92^n > 0.96$

$0.92^n < 0.04$

$n \log 0.92 < \log 0.04 \Rightarrow n > 38.6\ldots \Rightarrow n = 39$

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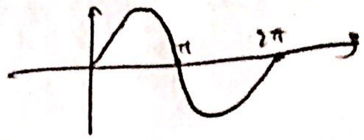
10. The curve C has equation $y = \sin\left(x + \frac{\pi}{4}\right)$, $\quad 0 \leqslant x \leqslant 2\pi$



(a) On the axes below, sketch the curve C. **(2)**

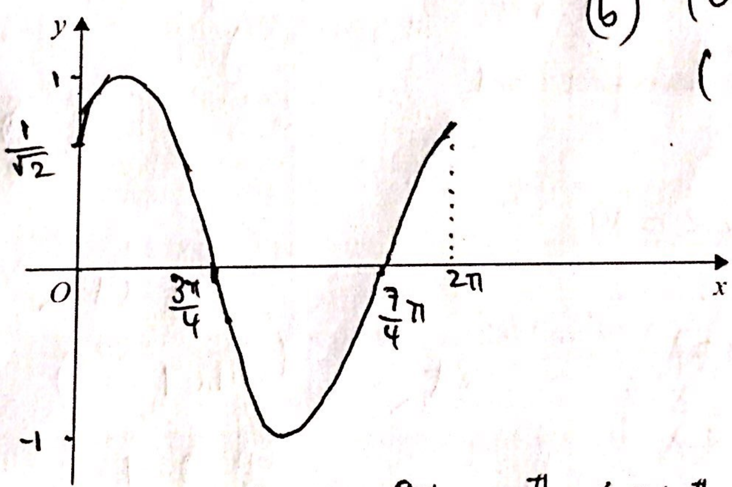
(b) Write down the exact coordinates of all the points at which the curve C meets or intersects the x-axis and the y-axis. **(3)**

(c) Solve, for $0 \leqslant x \leqslant 2\pi$, the equation

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$$

giving your answers in the form $k\pi$, where k is a rational number. **(4)**

(a)



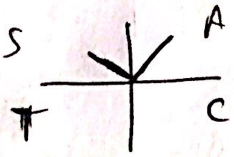
(b) $\left(0, \frac{1}{\sqrt{2}}\right)$

$\left(\frac{3\pi}{4}, 0\right)$

$\left(\frac{7\pi}{4}, 0\right)$

~~$\frac{\pi}{4} \le x+\frac{\pi}{4} \le \frac{9\pi}{4}$~~ $\qquad \frac{\pi}{4} \le x + \frac{\pi}{4} \le \frac{9\pi}{4}$

(c)



$\left(x + \frac{\pi}{4}\right) = \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3} \qquad x = \frac{1}{12}\pi^c$

$\left(x + \frac{\pi}{4}\right) = \frac{2\pi}{3} \qquad x = \frac{5}{12}\pi^c$

$\left(x + \frac{\pi}{4}\right) = \frac{7\pi}{3} \qquad \cancel{x = \frac{25}{12}\pi}$

X reject (outside range)

11.

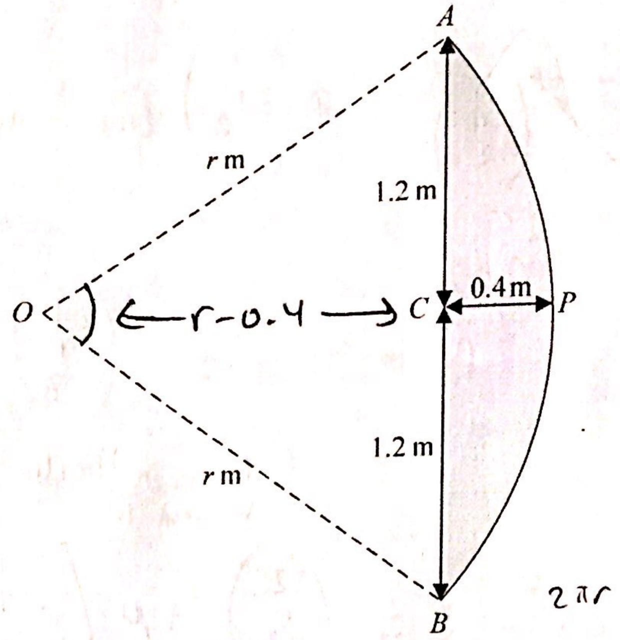


Diagram not drawn to scale

Figure 2

Figure 2 shows the design for a sail $APBCA$.

The curved edge APB of the sail is an arc of a circle centre O and radius r m.

The straight edge ACB is a chord of the circle.

The height AB of the sail is 2.4 m.

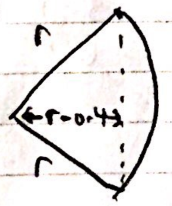
The maximum width CP of the sail is 0.4 m.

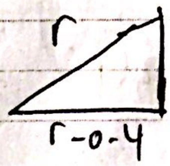
(a) Show that $r = 2$
(2)

(b) Show, to 4 decimal places, that angle AOB = 1.2870 radians.
(2)

(c) Hence calculate the area of the sail, giving your answer, in m^2, to 3 decimal places.
(4)

(a)





$\Rightarrow (r-0.4)^2 + 1.2^2 = r^2$

$\therefore\ r^2 - 0.8r + 1.6 = r^2$

$\therefore\ 0.8r = 1.6$

$\Rightarrow r = 2$ as required.

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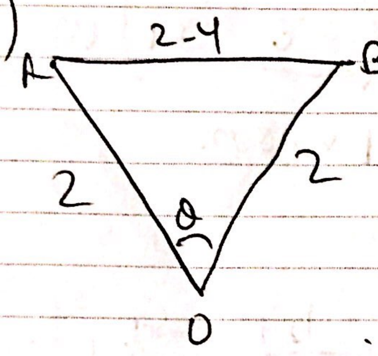
Question 11 continued

(b) ~~Arc length AB = ∠AÔB × 2²~~

(b)



$$\cos\theta = \frac{2^2 + 2^2 - 2.4^2}{2 \times 2 \times 2}$$

$$\therefore \cos\theta = \frac{7}{25}$$

$$\therefore \theta = \arccos\left(\frac{7}{25}\right) = 1.2870022\ldots$$

$$\therefore \angle AOB = 1.2870 \ (4dp)$$

(c) Area of seg = Area of sector OAB − Area Δ OAB

$$= \frac{1}{2}\arccos\left(\frac{7}{25}\right) \times 2^2 - \frac{1}{2} \times 2.4 \times 1.6$$

$$= 0.654 \ (3dp)$$

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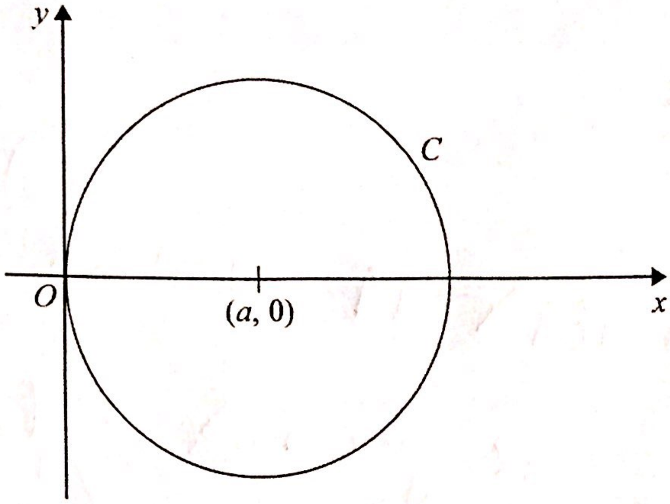

Figure 3

Figure 3 shows a circle C

C touches the y-axis and has centre at the point $(a, 0)$ where a is a positive constant.

(a) Write down an equation for C in terms of a

(2)

Given that the point $P(4, -3)$ lies on C,

(b) find the value of a

(3)

(a) $(x-a)^2 + y^2 = a^2$

(b) $(4-a)^2 + 9 = a^2$

$\therefore$ $\cancel{25+}$ $25 - 8a + a^2 = a^2$

$\therefore$ $a = \frac{25}{8}$

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13. (a) Show that the equation

$$2\log_2 y = 5 - \log_2 x \qquad x > 0, y > 0$$

may be written in the form $y^2 = \frac{k}{x}$ where k is a constant to be found.

(3)

(b) Hence, or otherwise, solve the simultaneous equations

$$2\log_2 y = 5 - \log_2 x$$

$$\log_x y = -3$$

for $x > 0, y > 0$

(5)

13(a). $2\log_2 y = 5 - \log_2 x$

$\therefore \quad 2\log_2 y + \log_2 x = 5$

$\therefore \quad \log_2 y^2 + \log_2 x = 5$

$\therefore \quad \log_2 (y^2 x) = 5$

$\therefore \quad \log_2 (y^2 x) = 5\log_2 2 = \log_2 2^5$

$= \log_2 32$

$\Rightarrow y^2 x = 32$

$\therefore \quad y^2 = \frac{32}{x} \qquad k = 32$

(b) $\log_x y = -3\log_x x = \log_x x^{-3} \Rightarrow y = \frac{1}{x^3}$

$\therefore \quad \cancel{\left(\frac{1}{x^3}\right)^2 = \frac{32}{x}}$

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$$y = \frac{1}{x^3} \quad \& \quad y^2 = \frac{32}{x}$$

$$\therefore \left(\frac{1}{x^3}\right)^2 = \frac{32}{x} \Rightarrow \frac{1}{x^6} = \frac{32}{x}$$

$$\therefore \cancel{1 = \frac{32}{x}}$$

$$1 = 32x^5 \Rightarrow x^5 = \frac{1}{32}$$

$$\therefore x = \frac{1}{2} \Rightarrow y = \frac{1}{\left(\frac{1}{2}\right)^3}$$

$$y = 8$$

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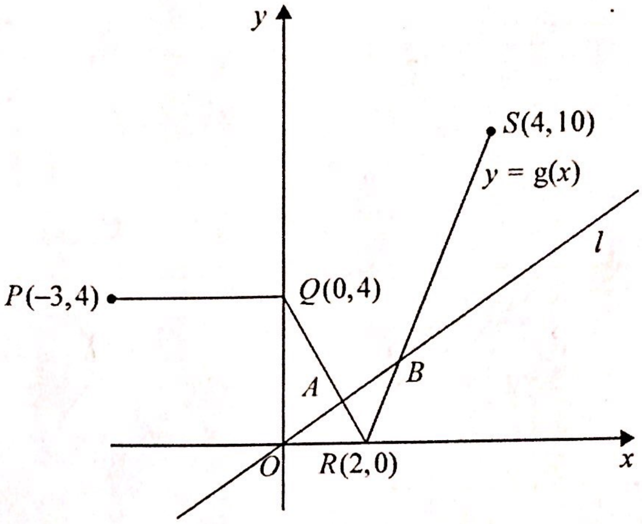

Figure 4

Figure 4 shows a sketch of the graph of $y = \mathrm{g}(x)$, $-3 \leqslant x \leqslant 4$ and part of the line l with equation $y = \frac{1}{2}x$

The graph of $y = \mathrm{g}(x)$ consists of three line segments, from $P(-3,4)$ to $Q(0,4)$, from $Q(0,4)$ to $R(2,0)$ and from $R(2,0)$ to $S(4,10)$.

The line l intersects $y = \mathrm{g}(x)$ at the points A and B as shown in Figure 4.

(a) Use algebra to find the x coordinate of the point A and the x coordinate of the point B.

Show each step of your working and give your answers as exact fractions.

(6)

(b) Sketch the graph with equation

$$y = \frac{3}{2}\mathrm{g}(x), \quad -3 \leqslant x \leqslant 4$$

On your sketch show the coordinates of the points to which P, Q, R and S are transformed.

(2)

(a) Consider QR: gradient $= -2$ $\quad c = 4$

$\Rightarrow y = -2x + 4$

$y = \frac{1}{2}x \Rightarrow \frac{1}{2}x = -2x + 4 \Rightarrow x_A = \frac{8}{5}$

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Consider SR: gradient = 5

$\therefore y = 5x + k$

$(2,0) \Rightarrow 0 = 10 + k \Rightarrow k = -10$

$y = 5x - 10$

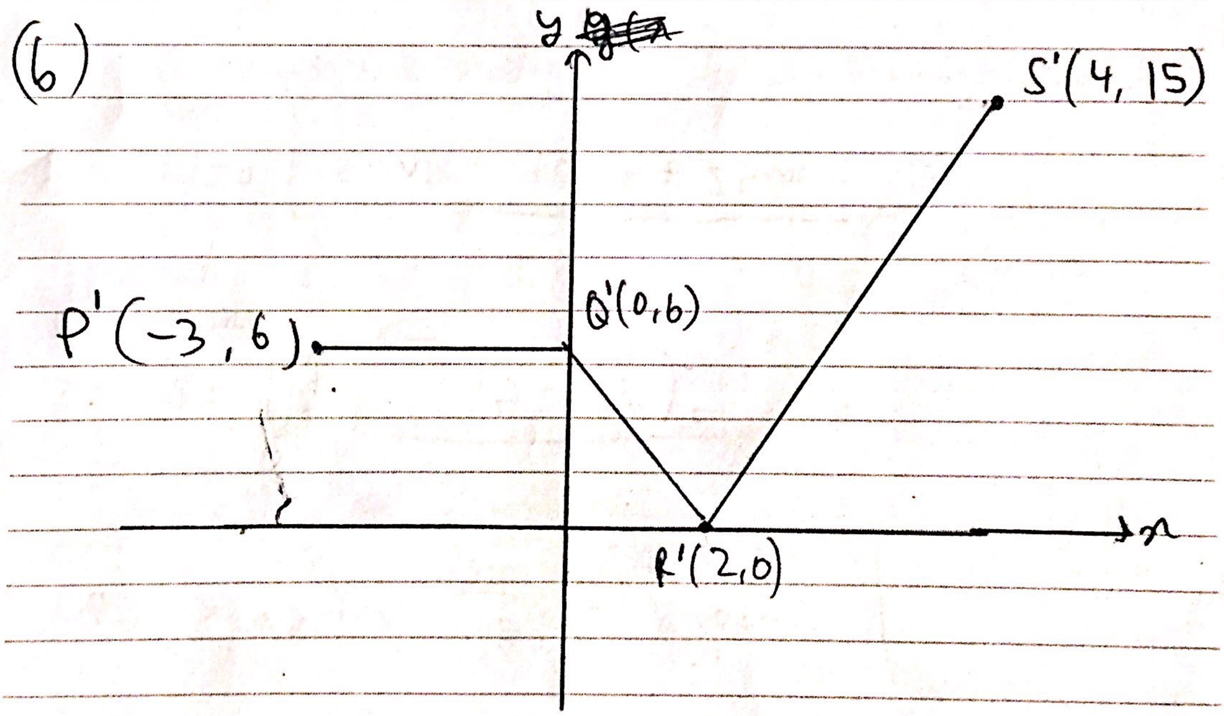
$y = \frac{1}{2}x \Rightarrow \frac{1}{2}x = 5x - 10$

$\frac{9}{2}x = 10$

$\therefore x_B = \frac{20}{9}$

$\therefore x_A = \frac{8}{5} \qquad x_B = \frac{20}{9}$

(b)



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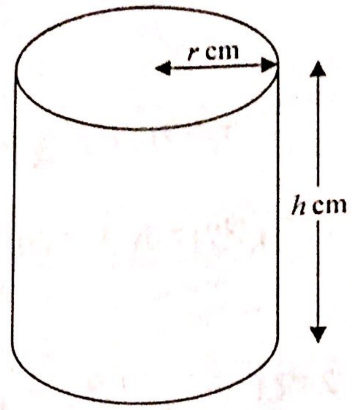


Figure 5

Figure 5 shows a design for a water barrel.

It is in the shape of a right circular cylinder with height h cm and radius r cm.

The barrel has a base but has no lid, is open at the top and is made of material of negligible thickness.

The barrel is designed to hold 60 000 cm^3 of water when full.

(a) Show that the total external surface area, S cm^2, of the barrel is given by the formula

$$S = \pi r^2 + \frac{120000}{r}$$

(3)

(b) Use calculus to find the minimum value of S, giving your answer to 3 significant figures.

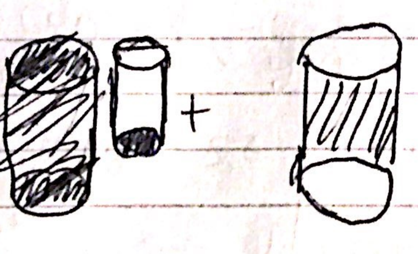
(6)

(c) Justify that the value of S you found in part (b) is a minimum.

(2)

(a) $V = \pi r^2 h = 60\,000 \Rightarrow h = \frac{60000}{\pi r^2}$

$S =$



$= \cancel{2}\pi r^2 + 2\pi r h$

$= \pi r^2 + 2\pi r\left(\frac{60000}{\pi r^2}\right)$

$= \pi r^2 + \frac{120\,000}{r}$ as required.

Question 5 continued

(b) $\frac{\partial S}{\partial r} = 2\pi r - 120000 r^{-2}$

$\frac{\partial s}{\partial r} = 0 \Rightarrow \cancel{2\pi r}(-1$

$2\pi r - \frac{120000}{r^2} = 0$

$\therefore \quad 2\pi r^3 - 120000 = 0$

$\therefore \quad r^3 = \frac{60000}{\pi}$

$\Rightarrow \quad r = 26.73\ldots \cancel{(3sf)}$

$\therefore S = \pi(26.73\ldots)^2 + \frac{120000}{26.73\ldots}$

$S = 6730 \ (3sf)$

(c) $\frac{\partial^2 S}{\partial r^2} = 2\pi + 240000 r^{-3}$

$\left(\frac{\partial^2 S}{\partial r^2}\right)_{r=26.73\ldots} = 2\pi + 240000(26.73\ldots)^{-3}$

$= 6\pi \ \cancel{\neq} > 0$

$\therefore \left(\frac{\partial^2 S}{\partial r^2}\right)_{26.73\ldots} > 0 \Rightarrow \underset{=6730}{S}$ is a minimum

Q5

(Total 5 marks)

16.

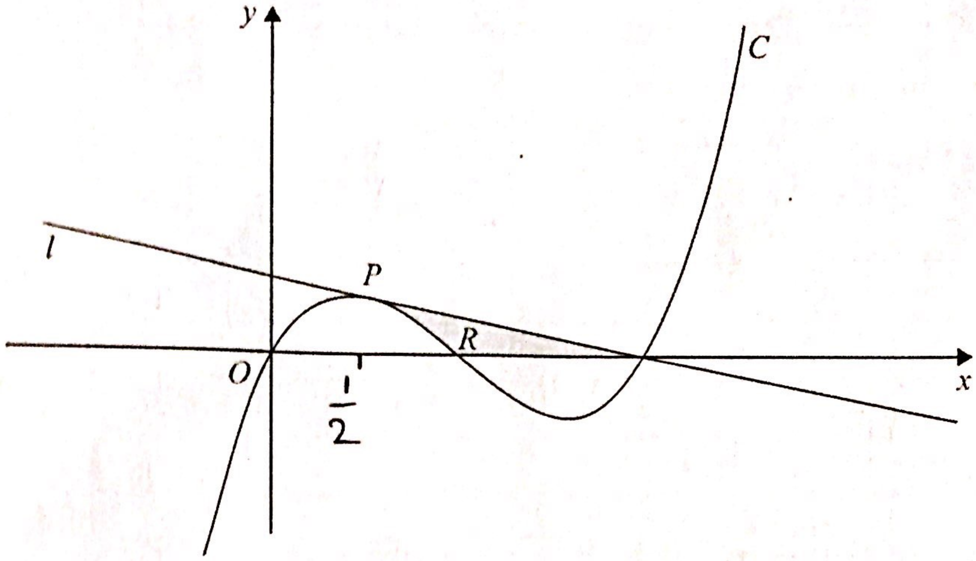


Figure 6

Figure 6 shows a sketch of part of the curve C with equation

$$y = x(x-1)(x-2)$$

The point P lies on C and has x coordinate $\frac{1}{2}$

The line l, as shown on Figure 6, is the tangent to C at P.

(a) Find $\frac{dy}{dx}$

(2)

(b) Use part (a) to find an equation for l in the form $ax + by = c$, where a, b and c are integers.

(4)

The finite region R, shown shaded in Figure 6, is bounded by the line l, the curve C and the x-axis.

The line l meets the curve again at the point (2, 0)

(c) Use integration to find the exact area of the shaded region R.

(6)

(a). $y = x^3 - 3x^2 + 2x$

$\frac{dy}{dx} = 3x^2 - 6x + 2$

(b) Gradient @ $P = \left(\frac{dy}{dx}\right)_{0.5} = -\frac{1}{4}$

Question 8 continued

$y_p = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)$

$= \frac{3}{8} \quad \therefore \; P\left(\frac{1}{2}, \frac{3}{8}\right)$

$\therefore y - y_p = -\frac{1}{4}(x - x_p)$

$\therefore y - \frac{3}{8} = -\frac{1}{4}\left(x - \frac{1}{2}\right)$

$\therefore 8y - 3 = -2x + 1$

$\therefore 2x + 8y - 4 = 0$

$\therefore x + 4y = 2$ ~~0~~

$a = 1$
$b = 4$
$c = 2$

(C) ~~Area R =~~ $\int$

Area R = $\triangle$ $- \int_{1/2}^{1} x^3 - 3x^2 + 2x \, dx$

$= \frac{1}{2} \times \frac{3}{2} \times \frac{3}{8} - \left[\frac{1}{4}x^4 - x^3 + x^2\right]_{1/2}^{1}$

$= \frac{9}{32} - \left[\left(\frac{1}{4}\right) - \left(\frac{9}{64}\right)\right]$

$= \frac{9}{32} - \frac{7}{64} = \frac{11}{64}$

Q8

(Total 12 marks)

Leave blank